Algebraic Manipulations Part II

1 Introduction

In math classes, you have been trained to solve systems using standard methods, such as substitution and elimination. However, in this first Math League lecture we will explore problems that requrire creativity and show that sometimes finding exact values of variables is not the best way to go. We will also explore a few advanced systems of equations that *do* ask to solve for the variables but utilized techniques explored in the first class of problems.

2 A Motivating Example

Here is an example of such a problem taken from last year's RHS Guts Round. The first solution solves for a and b, omitting some of the finer details. The second solution manipulates the givens in a more clever fashion.

Example 1 (AMSP Team Contest)**.** *Positive real numbers a and b satisfy*

$$a + \frac{1}{b} = 7 \quad \textit{and} \quad b + \frac{1}{a} = 5.$$

What is the value of $ab + \frac{1}{ab}$?

Solution. We first solve this sytem for a and b. Rearranging the second equation gives $b = 5 - \frac{1}{a}$. Substituting this expression for b into the first equation gives

$$a + \frac{1}{5 - \frac{1}{a}} = a + \frac{a}{5a - 1} = 7 \quad \Longrightarrow \quad a = \frac{1}{10}(35 \pm \sqrt{1085}).$$

WLOG let $a = \frac{1}{10}(35 + \sqrt{1085})$. (The solution when a is the other root of the above equation is similar.) Then

$$b = 5 - \frac{1}{a} = 5 - \frac{10}{35 + \sqrt{1085}} = \frac{1}{14}(35 + \sqrt{1085}).$$

This implies

$$ab = \left(\frac{35 + \sqrt{1085}}{10}\right)\left(\frac{35 + \sqrt{1085}}{14}\right) = \frac{33 + \sqrt{1085}}{2},$$

so finally we obtain

$$ab + \frac{1}{ab} = \frac{33 + \sqrt{1085}}{2} + \frac{2}{33 + \sqrt{1085}} = \boxed{33}.$$

■

OR

Solution. Multiply the two equations together to get

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 1 + 1 + \frac{1}{ab} = 35 \implies ab + \frac{1}{ab} = \boxed{33}.$$

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See the difference?

3 Many, Many Examples

Here are several examples that display certain techniques that are common in these types of problems. The first shows a common type of manipulation where we express everything in terms of $x + y$ and xy.

Example 2. *Let x and y be real numbers such that $x + y = 10$ and $xy = 11$. Determine the values of $x^2 + y^2$ and $(x + 1)(y + 1)$.*

Solution. For the first problem, we can square $x + y = 10$ to get

$$x^2 + 2xy + y^2 = (x^2 + y^2) + 2 \cdot 11 = 100 \implies x^2 + y^2 = \boxed{78}.$$

The second problem is even easier, as

$$(x + 1)(y + 1) = xy + x + y + 1 = 11 + 10 + 1 = \boxed{22}.$$

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As a general rule of thumb, whenever an expression is symmetric[1] it can be written in terms of simpler symmetric polynomials. This even extends to more complicated expressions; for example, we have

$$x^4 - 5x^3y - 5xy^3 + y^4 = (x + y)^4 - 9xy(x + y)^2 + 12(xy)^2.$$

Example 3. *Suppose x and y are real numbers such that $x + y = 3$ and $xy = 1$. Compute $x^4 - 5x^3y - 5xy^3 + y^4$.*

Solution. From the above rearrangement the answer is trivially $3^4 - 9 \cdot 1 \cdot 3^2 + 12 \cdot 1^2 = \boxed{12}$. ■

Sometimes, the simple one-step process of rewrite-then-plug-in doesn't always make for the simplest solution. Two or three steps may be necessary to make the algebra simpler.

Example 4. *Let a and b satisfy $a + b = 3$ and $a^2 + b^2 = 5$. What is $a^3 + b^3$?*

Solution. Note that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \implies a^3 + b^3 = (a + b)^3 - 3ab(a + b).$$

If we knew the value of ab, we could proceed to find the required answer. However, we *can* find this quantity! We utilize the sum of squares condition instead:

$$(a + b)^2 = a^2 + 2ab + b^2 \implies ab = \frac{(a + b)^2 - (a^2 + b^2)}{2} = \frac{3^2 - 5}{2} = 2.$$

Therefore

$$a^3 + b^3 = 3^3 - 3 \cdot 2 \cdot 3 = \boxed{9}.$$

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This technique of working with symmetric polynomials is also useful for solving systems of equations.

Example 5 (Math League HS 1980-1981)**.** *If $x < y$, find the ordered pair of real numbers (x, y) which satisfies*

$$x^3 + y^3 = 400 \qquad \text{and} \qquad x^2y + xy^2 = 200.$$

Solution. Note that

$$(x + y)^3 = x^3 + y^3 + 3(x^2y + xy^2) = 400 + 3 \cdot 200 = 1000 \implies x + y = 10.$$

Additionally from the second equation we have $x^2y + xy^2 = xy(x + y) = 10xy = 200$, so $xy = 20$. Now we have taken this original complicated system and reduced it to the simpler system

$$\begin{cases} x + y &= 10, \\ xy &= 20. \end{cases}$$

Solving this system (which can be done in many different ways) eventually gives the quadratic $t^2 - 10t + 20 = 0$, which has solutions $t = 5 \pm \sqrt{5}$. Therefore $(x, y) = \boxed{(5 - \sqrt{5}, 5 + \sqrt{5})}$. ■

We end with a couple of more interesting problems where standard manipulations are not enough.

Example 6 (Math League HS 1997-1998/2009-2010)**.** *In terms of x, what is the polynomial P of least degree, with integral coefficients, for which $P(\sqrt{3} + \sqrt{2}) = \sqrt{3} - \sqrt{2}$?*

[1] i.e. its value remains unchanged when two of the variables are swapped

Solution. Let $t = \sqrt{3} + \sqrt{2}$. Note that $\sqrt{3} - \sqrt{2} = \frac{1}{t}$, so we wish to find a polynomial $P(x)$ such that $P(t) = \frac{1}{t}$. Squaring gives

$$t^2 = (\sqrt{3} + \sqrt{2})^2 = 5 + \sqrt{24} \implies t^2 - 5 = \sqrt{24}.$$

Squaring again gives $(t^2 - 5)^2 = t^4 - 10t^2 + 25 = 24$, so $t^4 - 10t^2 = -1$ and $10t - t^3 = \frac{1}{t}$. The requested polynomial is $\boxed{10x - x^3}$. ■

Example 7. *Suppose a, b, and c are real numbers such that*

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right).$$

If $abc = 13$, what is $a + b + c$?

Solution. Let $s = a + b + c$. Expanding both sides of the equality and manipulating yields

$$\begin{aligned} abc + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} &= 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + \frac{1}{abc} \\ \implies abc + a + b + c &= 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \\ &= 1 + \frac{a+b+c}{abc} \\ \implies 13 + s &= 1 + \frac{s}{13}. \end{aligned}$$

Solving this equation yields $s = \boxed{-13}$. ■

4 Things to Remember

Many times, solving a problem of this type will boil down to recognizing one or several common factorizations and manipulations. Here are just a few; you will find some others in the exercises.

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$; $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$
- $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$

The most valuable technique, however, is simple: just try stuff! The answer will rarely pop out immediately; you'll often need to try different things and stumble a bit along the way. Even still, don't try things just at random - instead, focus on improving your position on the battlefield. This is where practice comes in, for it allows one to develop the kind of thinking that leads to these conclusions more quickly.

5 Practice Problems

1. Two warm-up problems.

   (a) [Math League HS 1991-1992] If $x + y = 5$ and $x - y = 1$, what is the value of $2^{x^2 - y^2}$?

   (b) [Math League HS 2008-2009] If $(x - 1)(y - 1) = 2008$, what is the value of $(1 - x)(1 - y)$?

2. Let p and q be positive real numbers such that $p + q = 7$ and $pq = 5$. Compute

   i) $p^2q + pq^2$

   ii) $p(1 - q) + q(1 - p)$

   iii) $p^3 + q^3$

   iv) $\dfrac{p}{q - 1} + \dfrac{q}{p - 1}$

3. Suppose x is a real number such that $x+\frac{1}{x}=\sqrt{2015}$. What is $x^2+\frac{1}{x^2}$?

4. [PSAT/NMSQT 2013] Let a and b be real numbers such that $a^2+b^2=20$ and $ab=8$. What is $\frac{(a+b)^2}{ab}$?

5. [AHSME 1987] If (x,y) is a solution to the system
$$xy=6 \qquad \text{and} \qquad x^2y+xy^2+x+y=63,$$
find x^2+y^2.

6. Mr. Douglas has a rectangular garden of math textbooks in his backyard with nonzero length and width and an area of 60 square feet. If he shortens both the length and the width of the book garden by one foot, its area decreases to 40 square feet. What will the area of the book garden be if the length and width are both shortened by one more foot?

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

7. [Math League HS 1998-1999] Suppose a and b are real numbers such that
$$\frac{1}{a(b+1)}+\frac{1}{b(a+1)}=\frac{1}{(a+1)(b+1)}.$$
What is $\frac{1}{a}+\frac{1}{b}$?

8. [AMC10 2000] Two non-zero real numbers, a and b, satisfy $ab=a-b$. Which of the following is a possible value of $\frac{a}{b}+\frac{b}{a}-ab$?

(A) -2 **(B)** $-\frac{1}{2}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** 2

9. [NIMO 2014, David Altizio] Let a and b be positive real numbers such that $ab=2$ and
$$\frac{a}{a+b^2}+\frac{b}{b+a^2}=\frac{7}{8}.$$
Find a^6+b^6.

10. [Mandelbrot 2013-2014] Let c be the larger solution to the equation $x^2-20x+13=0$. Compute the area of the circle with center (c,c) passing through the point $(13,7)$.

11. [Math League HS 1997-1998] In $\triangle ABC$, $\tan A$, $\tan B$, and $\tan C$ have integral values and form an increasing arithmetic progression. What is the value of $(\tan A)(\tan C)$?

★ 12. Two similar-looking Math League problems.

(a) [Math League HS 1982-1983] What are all ordered pairs of numbers (x,y) which satisfy
$$x^2-xy+y^2=7 \qquad \text{and} \qquad x-xy+y=-1?$$

(b) [Math League HS 1990-1991] What is the ordered pair of numbers (x,y), with $x>y$, for which
$$x^2+xy+y^2=84 \qquad \text{and} \qquad x+\sqrt{xy}+y=14?$$

★ 13. [HMMT 2004] Let x be a real number such that $x^3+4x=8$. Determine the value of x^7+64x^2.

★ 14. [AMSP Team Contest] Let a, b, c be nonzero numbers such that $a^2-b^2=bc$ and $b^2-c^2=ac$. Prove that $a^2-c^2=ab$.

★★ 15. [iTest 2008] Let a, b, c, and d be positive real numbers such that
$$\begin{aligned} a^2+b^2=c^2+d^2 &= 2008, \\ ac = bd &= 1000. \end{aligned}$$
If $S=a+b+c+d$, compute the value of $\lfloor S\rfloor$. (Note that $\lfloor x\rfloor$ denotes the greatest integer $\leq x$.)